Linear Algebra I

08/04/2020, Wednesday, 18:45 – 22:15

You are **NOT** allowed to use any type of calculators.

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1 Linear equations and column/null spaces (7 + 8 = 15 pts)

Let $A, B \in \mathbb{R}^{n\times n}$. Prove or disprove the following statements:

(a) There exists $X \in \mathbb{R}^{n\times n}$ such that $AX = B$ if and only if the column space of B is a subspace of the column space of A.

(b) There exists $Y \in \mathbb{R}^{n\times n}$ such that $YA = B$ if and only if the null space of A is a subspace of the null space of B.

2 Determinants (5 + 10 pts)

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$ be vectors such that $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = \|\boldsymbol{z}\| = \sqrt{3}$, $\boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{y}^T\boldsymbol{z} = 2$, and $\boldsymbol{x}^T\boldsymbol{z} = 1$. Also, let $A \in \mathbb{R}^{3\times 3}$ be a matrix such that

$$A\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}, \quad A\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{x}, \quad \text{and} \quad A\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}.$$

(a) Are $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ linearly dependent? Justify your answer.

(b) Find the determinant of A.

3 Partitioned matrices and diagonalizability (15 pts)

Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular and diagonalizable matrix. Is the matrix

$$\begin{bmatrix} M & M \\ M & M \end{bmatrix}$$

diagonalizable? Justify your answer.

4 Inverse matrix (7 + 8 = 15 pts)

Let $x \in \mathbb{R}^n$ be a nonzero vector and a be a real number. Determine all values of a such that $I_n - axx^T$ is nonsingular and find its inverse. (Hint: Its inverse is of the same form.)

5 Vectorspaces and linear transformations $((3+3+3)+(3+3)=15 \text{ pts})$

(a) Let S be a subspace of the vector space $\mathbb{R}^{n\times n}$.

(i) Show that the so-called *center* of S

$$C_S := \{A \in \mathbb{R}^{n\times n} \mid AX = XA \text{ for all } X \in S\}$$

is a subspace of $\mathbb{R}^{n\times n}$.

(ii) Let $n = 2$ and $S = \{A \mid A = A^T\}$. Determine the dimension of C_S.

(iii) Let $n = 2$ and $S = \{A \mid \operatorname{trace}(A) = 0\}$. Determine the dimension of C_S.

(b) Let x be a real number and $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be a transformation given by

$$\big(L(A)\big)_{ij} = \begin{cases} x & \text{if } i = j \\ a_{ij} & \text{if } i \neq j. \end{cases}$$

(i) Show that L is a linear transformation if and only if $x = 0$.

(ii) Let $n = 2$ and $x = 0$. Find the matrix representation of L relative to the ordered basis

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

6 Linear differential equations (15 pts)

Solve the linear differential equations

$$\begin{aligned} x_1'(t) &= 2x_1(t) - x_2(t) - x_3(t) \\ x_2'(t) &= -x_1(t) + 2x_2(t) - x_3(t) \\ x_3'(t) &= -x_1(t) - x_2(t) + 2x_3(t) \end{aligned}$$

with the initial conditions $x_1(0) = 0$, $x_2(0) = 1$, and $x_3(0) = 5$.

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10 pts free